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RADIO TECHNOLOGY AND THE THEORY OF NUMBERS *

BY

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The subject of my lecture combines two sciences which, at first sight, may perhaps seem utterly unrelated. In fact, I do not know of any literature which suggests a relationship between these two domains. I refer on the one hand to *Radio Technology* in its widest sense, a subject about which we have all heard a great deal and with which we are personally acquainted, if only through our use of radio sets at home; and on the other hand, to the *Theory of Numbers*, a part of pure mathematics, and perhaps even the purest part of it. Gauss, the great German mathematician, is said to have stated that "if mathematics is the Queen of Sciences, then Number Theory is the Queen of Mathematics." Number Theory is concerned for a large part with the natural numbers 1, 2, 3, $\cdots$, etc. These natural numbers have an unexpected wealth of properties, and some of the theorems discovered about them belong to the deepest regions human intellect has been able to penetrate. This led another German mathematician, Kronecker, to say "God created the integers, all the rest is the work of men."

We are, of course, to a certain extent, all acquainted with the natural numbers, for example, when in the morning we pay the milkman in Krone and Øre. But the natural numbers extend beyond a hundred and beyond a million, and even beyond the greatest numbers we encounter in astronomy. Hence the material at the disposal of those who study the Theory of Numbers extends towards infinity. And each of these natural numbers is either prime or composite. The primes are not divisible by any other integer apart from unity and the prime itself. Primes are, for instance, 2 or 3 or 5 or 7 or $\cdots$ 170,141,183,460,469,231,

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731,687,303,715,884,105,727, which latter number contains 39 digits when written in the usual decimal system.

It is, of course, not easy to verify that such a large number is not composite, that is, that it is not divisible by any smaller number except unity. And yet the American mathematician, Dr. D. H. Lehmer, told me recently at Los Angeles, when I visited his mathematical laboratory there, that he had discovered an even greater prime number. It certainly is the greatest number of which we are sure that it is indeed a prime. It is:

$$2^{1279} - 1$$

This number, written out in the usual decimal system, contains 397 digits. It took Dr. Lehmer's modern electrical digital computer 14 minutes to determine this number, whereas it would have taken a human being some 125 years to do so.

Now, what relation exists between Radio Technology and the Theory of Numbers?

Before I can answer this question, I would like to consider, in an even more general sense, the relation between mathematics on the one hand and physics and technology on the other. History shows us that mathematics has, to a great extent, been developed by pure mathematicians who were under the spell of the beauty of it; who were entranced by its mysterious generality and who spent their life discovering new facts and relations in this very wide and wonderful domain. Whether their results could be applied to astronomy, physics, chemistry, or say, technology, as a rule did not interest them primarily. Their main concern was to make their theorems cover cases of a more and more abstract nature, and generality in their work was their main purpose. In this sense we have to understand the famous saying of the English mathematician and philosopher, Sir Bertrand Russell, to the effect that "a mathematician is never so happy as when he does not know what he is talking about."

Some mathematicians even went as far as to be proud of the fact that none of their results could be applied in practice. It is said that one great mathematician remarked that "Bessel functions are beautiful functions in spite of their many applications."

Now, physics and technology were developed to a certain extent independently of pure mathematics. It is therefore natural that at a given moment the physicists or technologists discovered, to their surprise, that precisely the mathematical tools and methods they needed to solve their problems had already been fully developed by pure mathematicians who, at the time, did not have the slightest notion of, or even interest in, a possible practicable application of their results.

The main purpose of my lecture is to show that, in my view, both physics and radio technology have reached a stage where methods and

results properly belonging to the Theory of Numbers can be applied, and are beginning to be applied with much success, to the solution of problems typical of physics or radio technology.

I know that some mathematicians do not agree with this view. I hesitate to quote in this respect the great English mathematician, G. H. Hardy, and this for two reasons: first, because I have a great debt towards him for what he has given us in his beautiful mathematical researches, especially in the Theory of Numbers, and second, because he cannot contradict me any more. If I therefore venture to quote him, I do so with all due respect. In the otherwise charming and sometimes even pathetic essay (**1**) [2] which he wrote not long before his death, he says (pages 60–61):

> . . . If the Theory of Numbers could be employed for any practical and obviously honourable purpose, if it could be turned directly to the furtherance of human happiness or the relief of human suffering, as physiology and even chemistry can, then surely neither Gauss nor any other mathematician would have been so foolish as to decry or regret such applications. But science works for evil as well as for good (and particularly, of course, in time of war); and both Gauss and lesser mathematicians may be justified in rejoicing that there is one science at any rate, and that their own, whose very remoteness from ordinary human activities should keep it gentle and clean.

Again, another great mathematician and number theorist, Edmund Landau, wrote on page 21 of the first volume of his standard work (**2**): "Gordan pflegte etwa zu sagen: 'Die Zahlentheorie ist nützlich, weil man nämlich mit ihr promovieren kann'." [3]

These views may have been right and more or less complete at the time they were written, but it is my opinion that today physics and radio technology have reached a stage where ideas, methods and results, typically belonging to advanced Number Theory, can be applied with great success. Let me give you a few examples illustrating my thesis:

(*a*) In X-ray spectroscopy of cubic crystals it has been noted that certain reflections are absent. One finds reflections corresponding to the following integers: 1, 2, 3, 4, 5, 6, —, 8, 9, 10, 11, 12, 13, 14, —, 16, 17, $\cdots$ and it appeared that the missing numbers n are of the form $n = 4^k(8m + 7)$, where k and m are positive integers. Now Number Theory tells us that these are numbers which cannot be represented as the sum of three or less squares. There seems to be little doubt that the fact quoted is closely related to the three dimensions of our Euclidean space. I have inserted in Table I a short list of the natural numbers up to 28, with their representations as a sum of four or less squares. More than four squares are never required, but this number four is abso-

[2] The bold face numbers in parentheses refer to the references appended to this paper.

[3] "Gordan used to say: 'Number Theory is useful because it is a subject in which one may obtain a doctor's degree.' "

lutely necessary in the case of 7, 15, 23, $\cdots$, which are all of the form quoted above.

TABLE I.—*The Representation of the Natural Numbers by Four or Less Squares.*

$$
\begin{aligned}
1 &= 1^2\\
2 &= 1^2 + 1^2\\
3 &= 1^2 + 1^2 + 1^2\\
4 &= 2^2 = 1^2 + 1^2 + 1^2 + 1^2\\
5 &= 2^2 + 1^2\\
6 &= 2^2 + 1^2 + 1^2\\
7 &= 2^2 + 1^2 + 1^2 + 1^2\\
8 &= 2^2 + 2^2\\
9 &= 3^2 = 2^2 + 2^2 + 1^2\\
10 &= 3^2 + 1^2 = 2^2 + 2^2 + 1^2 + 1^2\\
11 &= 3^2 + 1^2 + 1^2\\
12 &= 2^2 + 2^2 + 2^2 = 3^2 + 1^2 + 1^2 + 1^2\\
13 &= 3^2 + 2^2 = 2^2 + 2^2 + 2^2 + 1^2\\
14 &= 3^2 + 2^2 + 1^2\\
15 &= 3^2 + 2^2 + 1^2 + 1^2\\
16 &= 4^2\\
17 &= 4^2 + 1^2 = 3^2 + 2^2 + 2^2\\
18 &= 3^2 + 3^2 = 4^2 + 1^2 + 1^2 = 3^2 + 2^2 + 2^2 + 1^2\\
19 &= 3^2 + 3^2 + 1^2 = 4^2 + 1^2 + 1^2 + 1^2\\
20 &= 4^2 + 2^2 = 3^2 + 3^2 + 1^2 + 1^2\\
21 &= 4^2 + 2^2 + 1^2\\
22 &= 3^2 + 3^2 + 2^2 = 4^2 + 2^2 + 1^2 + 1^2\\
23 &= 3^2 + 3^2 + 2^2 + 1^2\\
24 &= 4^2 + 2^2 + 2^2\\
25 &= 5^2 = 4^2 + 3^2 = 4^2 + 2^2 + 2^2 + 1^2\\
26 &= 5^2 + 1^2 = 4^2 + 3^2 + 1^2 = 3^2 + 3^2 + 2^2 + 2^2\\
27 &= 3^2 + 3^2 + 3^2 = 5^2 + 1^2 + 1^2 = 4^2 + 3^2 + 1^2 + 1^2\\
28 &= 4^2 + 2^2 + 2^2 + 2^2 = 3^2 + 3^2 + 3^2 + 1^2 = 5^2 + 1^2 + 1^2 + 1^2
\end{aligned}
$$

(*b*) The theory of potentials inside a crystal lattice (for example, the well known Madelung constant) can best be derived with the help of the elliptic theta-functions which at the same time form the basis of the analytical derivation in Number Theory of the representation of an integer as the sum of squares. In fact the two derivations are completely analogous.

(*c*) Several aspects of Planck's radiation formula and of the Einstein-Bose and Fermi statistics are closely related to Riemann's zeta-function $\zeta(s)$ which is defined by:

$$\zeta(s) = \frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \cdots = \frac{1}{\Gamma(s)} \int_0^\infty \frac{x^{s-1}}{e^x - 1}\, dx \qquad (\mathrm{Re}\, s > 1) \quad (1)$$

and which is at the base of practically all arithmetical investigations on the distribution of the prime numbers. This fact seems to have been realized in the United States to the extent that, during the war, even purely mathematical researches on the properties of the ζ-function were

classified as security subjects (verbal communication from Dr. Norbert Wiener).

(*d*) Another number-theoretic function, which is now beginning to play an important part in physics, is the partition function $p(n)$. It is defined by the generating function:

$$\prod_{k=1}^{\infty} \frac{1}{1-x^k} = 1 + \sum_{1}^{\infty} p(n)x^n. \qquad (|x| < 1) \tag{2}$$

The symbol $p(n)$ enumerates the number of ways in which the integer n can be written as the sum of any positive integers, repetitions being allowed. Thus, for example:

$$\begin{aligned} 6 &= 5+1 = 4+2 = 4+1+1 = 3+3 = 3+2+1 \\ &= 3+1+1+1 = 2+2+2 = 2+2+1+1 \\ &= 2+1+1+1+1 = 1+1+1+1+1+1 \end{aligned}$$

which shows eleven different ways of writing 6 as a sum of positive integers, and therefore $p(6) = 11$.

Applications of this partition function $p(n)$ and its asymptotic behavior are now being made in different parts of physics, among other things in the theory of the growth of crystals (3) and are used to explain the irregularities which sometimes occur on crystal surfaces.

These are a few examples borrowed from theoretical physics where modern problems can be successfully attacked with methods and results belonging to the field of Number Theory.

But also in radio fruitful applications of this most abstract part of mathematics can be made. Let me give you again a few instances.

(*e*) It is well known that when a periodic emf. is acting on a system which is generating *relaxation oscillations*, the system will, under certain circumstances, be forced to oscillate with a frequency which is a submultiple of the forcing frequency. Extensive use is made of these oscillations in television. Thus, we are confronted with a frequency demultiplication or frequency *division*. Well now, divisibility of numbers is a characteristic part of Number Theory, and perhaps it is even one of its most important chapters. It appears not improbable that here again arithmetical methods could help in further elucidating this highly complex phenomenon which is used extensively in practice. At the same time, several mathematicians of different nationalities are still pursuing theoretical research on this subject, among them Miss Mary L. Cartwright in Cambridge (England) (4).

(*f*) Let me give you next another example which has even a very classical flavor. It concerns the problem of the capacitance between two conducting spheres. A very elegant solution of this problem was given 100 years ago by Lord Kelvin with the aid of his theory of electrical images (5).

However, it is perhaps not generally known that Kelvin's expression for the capacitance $C_{a,b}$ of two mutually external spheres of radii a and b, respectively, and with their centers at a distance c apart, can be written in the form:

$$C_{a,b} = \frac{EI}{c} \sum_{n=1}^{\infty} \left\{ d(n) - d\left(\frac{n}{2}\right) \right\} \alpha^n, \tag{3}$$

where

$$\alpha = \frac{E - I}{E + I}$$

and E and I are the length of the external and internal tangents, respectively, to the circles obtained by cutting the sphere with a plane through their centers. In (3) we find the function $d(n)$, characteristic of Number Theory, representing the number of divisors of n. To take an example, $d(6) = 4$, because the number 6 has four divisors, *viz.* 1, 2, 3 and 6. Incidentally, in (3), $d\left(\frac{n}{2}\right)$ has to be replaced by zero for any odd n. Hence, in writing down some of the terms in (3) explicitly, we have:

$$C_{a,b} = \frac{EI}{c} \{ \alpha + \alpha^2 + 2\alpha^3 + \alpha^4 + 2\alpha^5 + 2\alpha^6 + 2\alpha^7 + \alpha^8 + 3\alpha^9 + 2\alpha^{10} + 2\alpha^{11} + 4\alpha^{12} + \cdots + 2\alpha^{6089} + 16\alpha^{6090} + 2\alpha^{6091} + \cdots \}. \tag{3a}$$

When it was found that $C_{a,b}$ could be written in the form (3) or (3a), with the coefficients $d(n)$ peculiar to Number Theory, the analogy of this problem with the famous Dirichlet problem was recognized. The latter consists of the calculation of the behavior of the sum $\sum_{1}^{N} d(n)$ when N tends towards infinity (6).

As far as we know the present literature does not contain any formulae giving an approximation for this capacitance $C_{a,b}$ when the spheres are very close together, that is, at a distance d which is small compared with both radii. In this case the variable α is close to unity and our series (3) or (3a) then converges very slowly indeed. The expression (3), however, enabled us to apply mathematical methods which are peculiar to Number Theory. Thus in the case of two spheres of equal radii a and a small distance d apart the following simple expression was found for their mutual capacitance:

$$C_{a,a} \approx \tfrac{1}{4} a \log \left(\frac{a}{d}\right). \tag{4}$$

A complete derivation of (4) is given in the Appendix.

Thus we see how, even in a very classical electrical problem, analytical methods borrowed from modern Number Theory may be used

(*g*) But there is another typical—and very fundamental—radio problem which I have never seen plainly explained in the literature and which also leads us to borrow results from Number Theory.

Suppose a voltage v consisting of the superposition of two cosine functions of time, and which is therefore of the form

$$v = a \cos \omega_1 t + b \cos \omega_2 t, \tag{5}$$

is applied to a detector with a square law characteristic representable by

$$i = 2\alpha v^2. \tag{6}$$

Substitution of (5) into (6) leads to:

$$i = \alpha\{a^2(1 + \cos 2\omega_1 t) + b^2(1 + \cos 2\omega_2 t) \\ + 2ab\{\cos (\omega_1 + \omega_2)t + \cos (\omega_1 - \omega_2)t)\}.$$

This formula clearly shows that the following frequencies are present in the current:

$$0, \quad 2\omega_1, \quad 2\omega_2, \quad \omega_1 + \omega_2, \quad \omega_1 - \omega_2.$$

When, however, the detector characteristic is such that it is only representable by a more complicated transcendental function than a parabola, such as is the case with the so-called "linear" detection, or when $i = e^{\beta v}$, then the frequencies—including the negative ones—present in the resulting current, generally comprise all possible combination tones, *viz.*:

$$m\omega_1 + n\omega_2. \tag{7}$$

Here both m and n run independently through all positive and negative integers, including zero. When further ω_1 and ω_2 are commensurate (which is always at least approximately true), and using a proper time scale, (7) can be replaced by:

$$ma + nb,$$

where now a and b are integers. A combination of integers such as expressed by (7*a*) is written $[a, b]$ and is called in Number Theory a "modulus" (7). Incidentally, in connection with this, Fueter quotes a book by Wilhelm Fliess, entitled "Zum Ablauf des Lebens," in which the latter maintains that the modulus [23, 28] represents all the numbers important in human life. Fliess shows, for example, that:

$$\begin{aligned} 1 \times 28 - 1 \times 23 &= 5 \\ 3 \times 23 - 2 \times 28 &= 13 \\ 5 \times 23 - 4 \times 28 &= 3 \\ 5 \times 28 - 6 \times 23 &= 2 \end{aligned}$$

the resulting numbers 5, 13, 3, 2 all representing ages of special importance in human life. However, as Fueter clearly points out, the numbers represented by the modulus [23, 28] or the numbers

$$23n + 28m,$$

where m and n run through all integers positive and negative, in point of fact cover every whole number from minus infinity to plus infinity. All we have to do, to represent any given integer, is to choose a proper combination of m and n.

This is the case for the base 23 and 28 because the highest common divisor of 23 and 28 is unity. If we had taken, for example, the modulus

$$16n+20m$$

the numbers generated by this formula are obviously all multiples of 4 because the highest common divisor of 16 and 20 is 4.

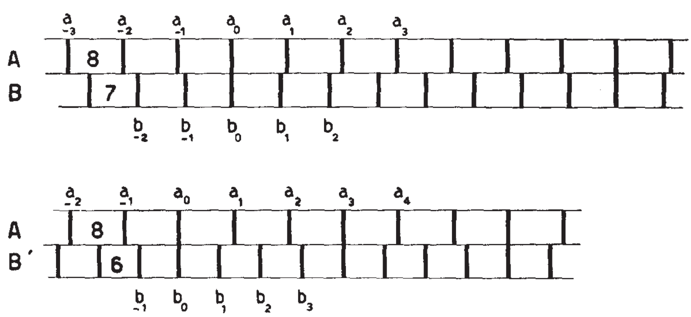

FIG. 1. Vernier showing the distribution of the combination tones when two sinusoidal oscillations are applied to a detector.

Returning now to our radio problems of the combination tones, as given by (7*a*), the general theory tells us that the modulus (7*a*) represents *all positive and negative multiples of the highest common divisor of the numbers* a *and* b, or (with a proper transformation of the time scale) of ω_1 and ω_2. Therefore the combination tones obtained with a transcendental detector are all nicely and equally spaced at a frequency distance corresponding to the highest common divisor of a and b, provided ω_1 and ω_2 are commensurate, otherwise the combination tones are "dense everywhere." We can thus, for example, state beforehand how many combination tones there will be found in a given frequency band, this number depending only on the width of the band and not on its position in the frequency spectrum.

The theorem mentioned, *viz.* that the numbers generated by

$$ma + nb, \qquad (7a)$$

where a and b are fixed integers, but m and n run, independently, through all positive and negative integers, are nothing else than the multiples of the highest common divisor of two numbers a and b, is of very fundamental importance in Number Theory. In fact, the highest common divisor of two numbers a and b can best be defined as the minimum positive value of the expression (7a) obtainable by a proper selection of m and n.

It is of interest to note here that this fundamental theorem is related to what is known in technology as the *vernier* or *nonius*. It is best explained with the aid of two linear scales. In Fig. 1 we have two scales A and B. The distance between successive markings (a_n, a_{n+1}) on scale A is 8 units, and that between successive markings (b_m, b_{m+1}) on scale B is 7 units. Now, we first place the scales in such a way that the—arbitrarily chosen—zero marks (a_0 and b_0) coincide. Then, it will be clear that the distance between any mark a_n on scale A and any other mark b_m on scale B is

$$8n - 7m \text{ units.}$$

In particular, we see clearly from the figure that

$$\begin{aligned} a_1 - b_1 &= 1 \text{ unit} \\ a_2 - b_2 &= 2 \text{ units} \\ a_3 - b_3 &= 3 \text{ units} \\ &\cdots\cdots\cdots\cdots \end{aligned}$$

and hence all integers can thus be generated. The smallest distance, except zero (here 1) is the highest common divisor of *8* and *7*.

Consider next Fig. 1a where we have again scale A, while scale B is replaced by scale B', in which the markings are six units apart.

The smallest distance (apart from zero) of a marking on scale A and of one on scale B' is now 2, for example, between a_1 and b_1, or between b_3 and a_2. All the other distances are multiples of 2, which is itself the highest common divisor of *8* and *6*.

We now come to our last example taken from radio technology.

(h) It is the so-called "sawtooth" function, which is depicted in Fig. 2. We denote it by Sa(x). It reaches its highest values $\frac{1}{2}$ (and at

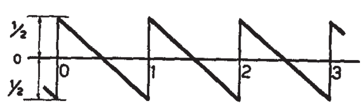

FIG. 2. The sawtooth function $y = \text{Sa}(x)$.

the same time its lowest values $-\frac{1}{2}$) at $x = \cdots - 3, -2, -1, 0, 1, 2, 3, \cdots$ and therefore shows a jump of magnitude unity at the integer points, whereupon every time it falls linearly.

This function of time is used in any television transmitter and receiver in order to scan periodically the lines and frames of the picture. Important technical developments have recently been achieved to generate sawtooth functions of proper shape and behavior experimentally. But, at the same time, this very sawtooth function $\mathrm{Sa}(x)$, which we use so extensively in radio technology, is the most fundamental function in Number Theory. This may be clarified with the

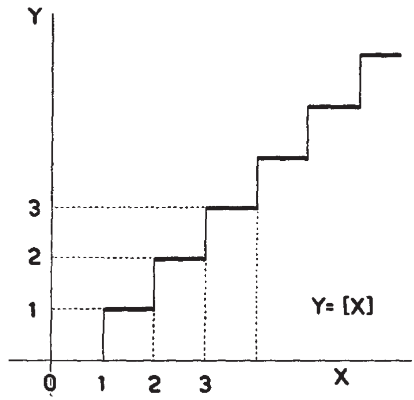

FIG. 3. Graph of the function $y = [x]$ representing the integers.

aid of a very simple construction. When we represent the integers graphically we obtain Fig. 3. This staircase function is denoted in Number Theory by

$$y = [x]. \tag{8}$$

If we subtract from (8) the simple function $y = x$ and add $\frac{1}{2}$, we obtain at once the sawtooth function $\mathrm{Sa}(x)$

$$\mathrm{Sa}(x) = [x] - x + \tfrac{1}{2}, \tag{9}$$

which is periodic with a period unity, and therefore has the property

$$\mathrm{Sa}(x+1) = \mathrm{Sa}(x).$$

This sawtooth function $\mathrm{Sa}(x)$ is so fundamental in Number Theory that it is not surprising that it occurs in it very often indeed. May I give you a few examples:

The first one concerns Stirling's formula for the factorial

$$\Pi(x) = \Gamma(x+1).$$

We have the well known expression:

$$\Pi(x) = \sqrt{2\pi x}\, x^x e^{-x} e^{\mu(x)} \qquad (x > 0), \tag{10}$$

where $\mu(x)$ tends towards zero for $x \to \infty$. This latter, rather complicated, function $\mu(x)$ has a very elegant and simple representation in terms of the sawtooth function; it is:

$$\mu(x) = \int_0^\infty \frac{\mathrm{Sa}(u)}{u + x}\, du. \tag{11}$$

But (11) can be generalized further if we introduce Hurwitz's generalization (8) $\zeta(s, x)$ of the Riemann zeta-function $\zeta(s)$. It is defined, for Re $s > 1$, by

$$\zeta(s, x) = \sum_{n=0}^{\infty} \frac{1}{(n + x)^s} \qquad \left.\begin{array}{r} \mathrm{Re}\, s > 1 \\ x > 0 \end{array}\right\}$$

so that

$$\zeta(s, 1) = \zeta(s).$$

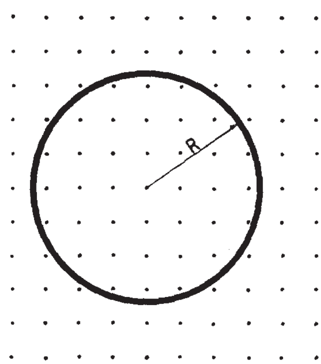

FIG. 4. The number of lattice points within a circle of radius R.

The function $\zeta(s, x)$ can be analytically continued for Re $s < 1$, and it can again be represented with the aid of the sawtooth function Sa(x) as follows:

$$\frac{1}{s}\left\{\zeta(s, x + 1) - \frac{1}{(s - 1)x^{s-1}} + \frac{1}{2x^s}\right\} = \int_0^\infty \frac{\mathrm{Sa}\, u}{(u + x)^{s+1}}\, du \qquad \left.\begin{array}{r} \mathrm{Re}\, s > -1 \\ x > 0 \end{array}\right\}. \tag{12}$$

Thus (12) is a generalization of (11) to which it is reduced if in (12) we take $s = 0$.

Again, if in (12) we restrict s to the range $-1 < \mathrm{Re}\, s < 0$, x may be taken zero, and we thus obtain for the Riemann's zeta function the

elegant expression:

$$\frac{1}{s}\zeta(s) = \int_0^\infty \frac{\mathrm{Sa}(u)}{u^{s+1}}\,du, \qquad -1 < \mathrm{Re}\, s < 0 \tag{13}$$

a function to the study of which so much has been contributed by the great Danish mathematician Harold Bohr, and which is at the basis of the whole theory of the distribution of prime numbers. The expressions (11), (12) and (13) give only a few examples of the occurrence of the sawtooth function in analysis.

Another example, typical of Number Theory, is the following:

Consider a square lattice as depicted in Fig. 4, where the distance between different points is taken as unity. If we draw a circle, with its center at any lattice point and with a radius of length R, the number

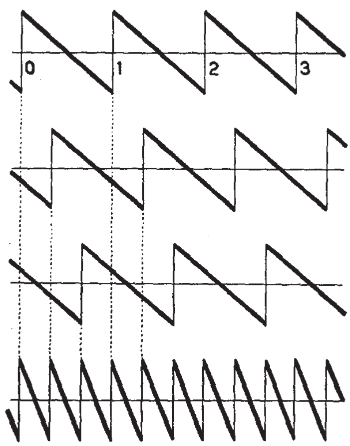

FIG. 5. The addition of three sawtooth functions shifted over $\frac{0}{3}$, $\frac{1}{3}$ and $\frac{2}{3}$ of a period, respectively, yield another sawtooth function having three times the original frequency.

of lattice points within the circle will obviously increase with R. A classical problem in Number Theory is to calculate the number $A(R^2)$ of lattice points within a circle of radius R. It will be clear at the outset that, if R becomes very large, this number will approximately equal the area of the circle, that is, πR^2. The correction to be applied to this approximation in order to obtain the exact number can be expressed elegantly as follows in series of sawtooth functions, where $R = \sqrt{x}$:

$$\tfrac{1}{4}\{A(x) - \pi x\} = \mathrm{Sa}(x) - \mathrm{Sa}\left(\frac{x}{3}\right) + \mathrm{Sa}\left(\frac{x}{5}\right) - \mathrm{Sa}\left(\frac{x}{7}\right) + \cdots.$$

The sawtooth function $\mathrm{Sa}(x)$ also has properties which are visible at once from a graphical construction. In fact, the periodic sawtooth

function $\mathrm{Sa}(x)$ is in some respects simpler than, say, the sine function $\sin 2\pi x$, although its Fourier representation

$$\mathrm{Sa}(x) = \sum_{k=-\infty}^{k=+\infty}{}' \frac{e^{2\pi ikx}}{2\pi ik} = \sum_{k=1}^{\infty} \frac{\sin 2\pi kx}{\pi k} \tag{14}$$

contains an infinity of sine terms.

One of the properties referred to is:

$$\sum_{k=1}^{n} \mathrm{Sa}\left(x - \frac{k}{n}\right) = \mathrm{Sa}(nx), \tag{15}$$

which is shown in Fig. 5, where $n = 3$. Expressed in words, this means that the superposition of n sawtooth functions, each one shifted over a phase difference $1/n$ with respect to the preceding one, yields a further

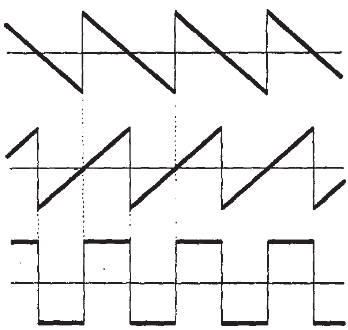

FIG. 6. The subtraction from a sawtooth function of another sawtooth function shifted over half a period gives a "square sine" function.

sawtooth function, but of n times the frequency. Properties of this sort surely have applications in radio practice. There are also other properties of the sawtooth function (see Fig. 6), which might be used experimentally, such as:

$$\mathrm{Sa}(x) - \mathrm{Sa}(x - \tfrac{1}{2}) = \tfrac{1}{2}\mathrm{Sin}\,(2\pi x), \tag{16}$$

where $\mathrm{Sin}\,(2\pi x)$ represents the "square sine" function which jumps from -1 to $+1$ at $x = \cdots, -1, 0, 1, 2, 3, \cdots$ and from $+1$ to -1 at $x = \cdots, -\frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \cdots$ whereas it stays constant at 1 or -1 in the intervals between the jumps. It can also be written in the two forms

$$\mathrm{Sin}\,(2\pi x) = \frac{\sin 2\pi x}{|\sin 2\pi x|} = (-1)^{[2x]}.$$

It therefore has some analogy with the ordinary sine function, except that it behaves quite discontinuously.

It is not difficult, once again, to show (for example, graphically) that there exists also the following way (Fig. 7) to produce a $\text{Sin}\ (2\pi x)$

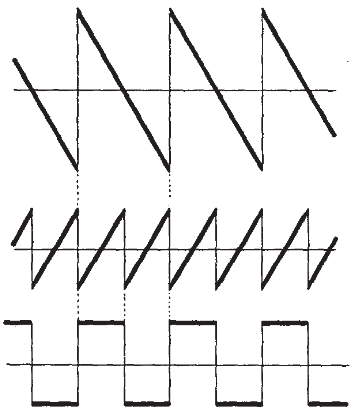

FIG. 7. The subtraction from a sawtooth function of another sawtooth function of half the amplitude but of double the frequency yields a "square sine" function.

function from sawtooth functions:

$$2\ \text{Sa}(x) - \text{Sa}(2x) = \tfrac{1}{2}\text{Sin}\ (2\pi x). \tag{17}$$

Incidentally, (17) follows from (16) and (15), when, in the latter, we take $n = 2$. All the above properties of the sawtooth functions and many others can most easily be derived with the help of the operational calculus (9).

It can also be shown that $\text{Sa}(x)$ can be developed in a series of square sine functions of frequency 2^n. In fact, we have:

$$\text{Sa}(x) = \sum_{n=1}^{\infty} \frac{1}{2^{n+1}} \cdot \text{Sin}\ (2^{n-1} \cdot 2\pi x) = \sum_{1}^{\infty} \frac{1}{2^{n+1}} (-1)^{[2^n x]}.$$

This synthesis of the sawtooth function $\text{Sa}(x)$ from square sine functions is illustrated in Fig. 8. At the top we have five square sine functions a, b, c, d and e. The graph A is the synthesis of a and b, while B is the synthesis of a, b, c, d and e.

This development of the $\text{Sa}(x)$ function has, contrary to its Fourier development (14), the interesting property that the so-called Gibbs' phenomenon is absent here. A similar synthesis of a sawtooth function from its ordinary Fourier components (sine functions) with the resulting Gibbs' phenomenon is shown in Fig. 9, where the "overshoots" at the discontinuities are clearly visible.

I also recently found the following functional relations of the sawtooth function:

$$\text{Sa}(\text{Sa}\ x) = -\ \text{Sa}(x + \tfrac{1}{2})$$

and therefore:

$$\mathrm{Sa}\{\mathrm{Sa}(\mathrm{Sa}(x))\} = \mathrm{Sa}(x). \tag{18}$$

Thus (18) shows us that a sawtooth of a sawtooth of a sawtooth is again the original sawtooth function. Therefore any odd number of iterations always reproduces the original function.

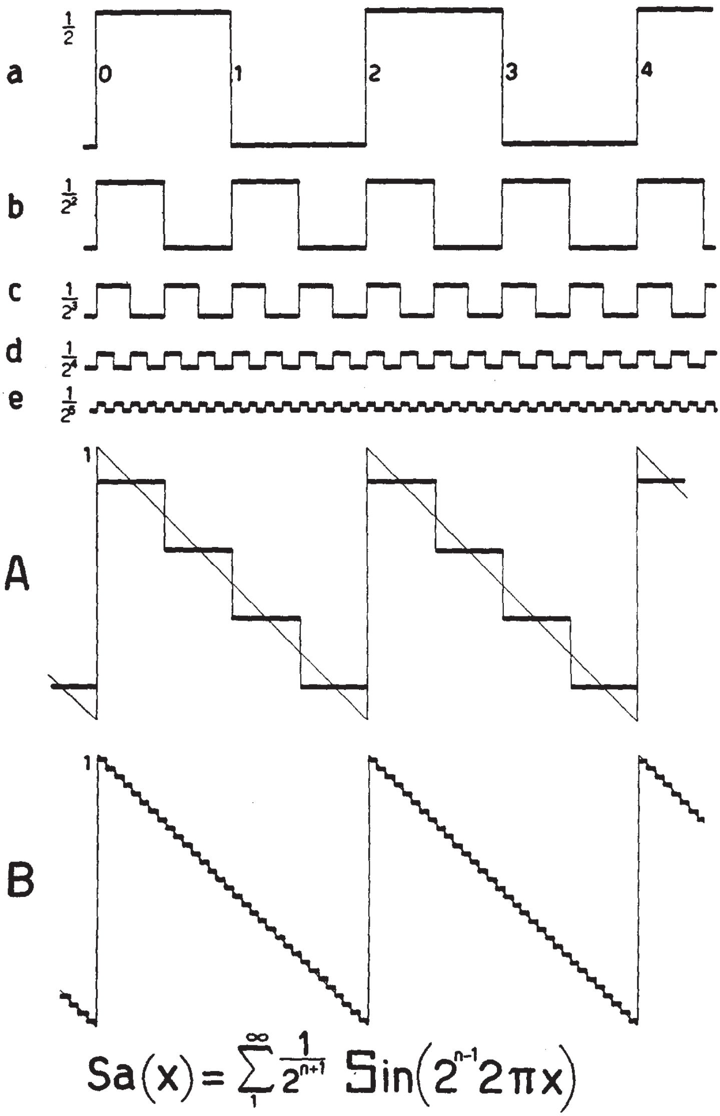

FIG. 8. The synthesis of a sawtooth function from "square sine" functions.

The relations expressed by (15), (16), (17) and (18) are all relatively simple, but there are much deeper relations involving the sawtooth function. For instance, Landau (Ref. 2, II, 170), proves the following interesting formula:

$$\int_0^1 \mathrm{Sa}(mx)\cdot\mathrm{Sa}(nx)dx = \frac{1}{12}\frac{(m, n)}{\{m, n\}}. \tag{19}$$

Here m and n represent two positive integers and (m, n) stands for the highest common divisor of m and n, while $\{m, n\}$ represents their least common multiple.

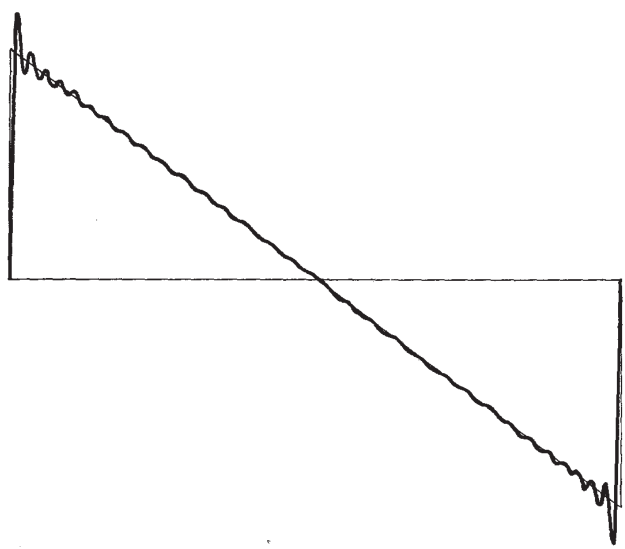

FIG. 9. The Fourier synthesis of a sawtooth function from sine functions shows the typical "overshoot" at the discontinuities ("Gibbs' phenomenon").

This is a typical and rather deep lying example of the occurrence of the sawtooth function in Number Theory.

Finally, properties such as those expressed by (15), (16), (17) and (19) may be further generalized if we note that $\mathrm{Sa}(x)$ can be identified with the negative of the first Bernoullian polynomial $B_1(x) = x - \frac{1}{2}$, when the latter is extended periodically outside the range $0 < x \leq 1$.

Therefore, we have:

$$\mathrm{Sa}(x) = -B_1(x - [x]). \tag{20}$$

In fact we can define the Bernoullian polynomials $B_n(x)$ with the aid of the generating function:

$$\frac{te^{tx}}{e^t - 1} = \sum_0^\infty \frac{B_n(x)}{n!} t^n.$$

Known properties of the polynomials $B_n(x)$ can therefore be applied at once to our sawtooth function $\mathrm{Sa}(x)$, and a few of these may have some technical value. I recently obtained a formula which generalizes Landau's relation (19). I will not quote its most general form here, but the following is a special case:

$$\int_0^1 B_k(mx - [mx]) \cdot B_k(nx - [nx]) \cdot dx$$

$$= (-1)^{k+1} \cdot \frac{k!k!}{(2k)!} B_{2k}(O) \left(\frac{(m, n)}{\{m, n\}} \right)^k. \quad (21)$$

In this formula we therefore obtain the kth power (k = integer) of the ratio $\frac{(m, n)}{\{m, n\}}$ (the quotient of the highest common divisor of m and n, and their least common multiple) instead of the first power, as in Landau's formula (19).

I hope that the above arguments and examples may show that time is ripe indeed for applying to physics and radio technology, methods and results obtained in what may perhaps be called the most abstract and the purest branch of mathematics, the Theory of Numbers.

I myself do not feel that these technical applications will rob it of any of its manifold beauties, charms and intricacies. On the contrary, more of its—still hidden—theorems and relations may thus be brought to light. In fact, technology in its turn is already in the process of assisting Number Theory to a considerable extent, now that, for example, the numbers e and π, have been calculated to over 2000 decimal places with the aid of modern electrical digital computers. Methods are also on the way and plans have been worked out to test numerically, with the same apparatus, the famous Riemann hypothesis which conjectures that all the non-trivial zeros of the function $\zeta(s)$ lie on the line $\mathrm{Re}\, s = \frac{1}{2}$. This hypothesis is of the utmost importance for modern Number Theory and many results in Higher Arithmetic could be considerably refined if the truth of it could be proved. This problem however, is still unsolved today, and so far it has defied the most ingenious, intricate and elaborate methods of modern analysis, even when applied by the greatest among modern mathematicians. Technology may—perhaps very soon—produce some decisive contributions in this most interesting field. It can certainly be said that the whole of Num-

ber Theory would exhibit quite a different aspect if, with modern electrical methods, one single non-trivial zero of $\zeta(s)$, out of the infinite number that we know exist, were to be found elsewhere than on the line $\text{Re } s = \frac{1}{2}$.

Harmonious collaboration between pure mathematicians and trained technologists is now beginning to yield far reaching results of fundamental importance in Number Theory, as in so many other branches of science. A case in point is the Mathematical Center in Amsterdam, where the computation department—amply equipped with electrical and mechanical computers—forms a valuable and stimulating support for those engaged in abstract research, both inside and outside this institute.

These are the lines along which, I feel, we may expect great progress in this almost unlimited field, full as it is of the most beautiful relations, absolutely peculiar to it and for which we look in vain in other branches of mathematics—the field of The Theory of Numbers.

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APPENDIX

The result of the calculation, with the help of Kelvin images, of the capacitance $C_{a,b}$ between two mutually external spheres of radius a and b, respectively, having their centers a distance c apart, can be written in the form:

$$C_{a,b} = \frac{EI}{c} \sum_{n=1}^{\infty} \frac{\alpha^n}{1 - \alpha^{2n}} = \frac{EI}{c} h(\alpha), \qquad \text{say (10)} \tag{1}$$

where

$$\alpha = \frac{E - I}{E + I},$$

while

$$E^2 = c^2 - (a - b)^2$$

and

$$I^2 = c^2 - (a + b)^2,$$

E and I representing the lengths of the external and internal tangents to the two great circles of the spheres in a plane through their centers.

Now, the series in (1) can be transformed as follows:

$$h(\alpha) = \sum_{n=1}^{\infty} \frac{\alpha^n}{1 - \alpha^{2n}} = \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} \alpha^{(2m+1)n}$$

which expression leads at once to the following:

$$h(\alpha) = \sum_{n=1}^{\infty} \left(d(n) - d\left(\frac{n}{2}\right) \right) \alpha^n, \tag{2}$$

where $d(n)$ is the number of divisors of the number n (for example, $d(6) = 4$ because 6 has the four divisors 1, 2, 3 and 6), and where $d\left(\frac{n}{2}\right)$ has to be replaced by zero when n is odd.

The series of (2) is closely related to the famous divisor problem of Dirichlet (Ref. 6, p. 262).

In order to derive an approximation for (2) for the case where the distance between the spheres is small compared with their radii (corresponding to α being just below unity), we can proceed as follows.

We introduce the new variable x through the relation

$$\alpha = e^{-e^{-x}}$$

so that (2) becomes

$$h(e^{-e^{-x}}) = \sum_{n=1}^{\infty} \left\{ d(n) - d\left(\frac{n}{2}\right) \right\} e^{-ne^{-x}} = \sum_{n=1}^{\infty} d(n) \{ e^{-ne^{-x}} - e^{-2ne^{-x}} \}. \tag{3}$$

The operational image (9) of $e^{-e^{-x}}$ is $\Pi(p)$ or

$$e^{-e^{-x}} \doteqdot \Pi(p), \qquad (\operatorname{Re} p > 0)$$

so that we obtain

$$h(e^{-e^{-x}}) \doteqdot \left(1 - \frac{1}{2^p}\right) \Pi(p) \cdot \zeta^2(p). \qquad (\operatorname{Re} p > 1)$$

An approximation of (3) for $x \to \infty$, is given by the residue at $p = 1$ of the inversion integral

$$\begin{aligned} h(e^{-e^{-x}}) &\approx \frac{1}{2\pi i}\oint_{p=1}\left(1 - \frac{1}{2^p}\right)\frac{\Pi(p)}{p}\zeta^2(p)e^{px}dp \\ &= \frac{1}{2\pi i}\oint_{p=1}\left(1 - \frac{1}{2^p}\right)\frac{\Pi(p)}{p}\left(\frac{1}{p-1} + \gamma + \cdots\right) \\ &\qquad\qquad \times \left(\frac{1}{p-1} + \gamma + \cdots\right)e^{px}dp \end{aligned}$$

(where $\gamma = 0.57722\cdots$ is Euler's constant), or, with $p = 1 + z$,

$$\begin{aligned} &= \frac{e^x}{2\pi i}\oint_{z=0}\left(1 - \frac{1}{2^{z+1}}\right)\Pi(z)\left(\frac{1}{z} + \gamma + \cdots\right)\left(\frac{1}{z} + \gamma + \cdots\right)e^{zx}dz \\ &= \frac{e^x}{2\pi i}\oint_{z=0}\tfrac{1}{2}(1 + z\log 2 + \cdots)(1 - \gamma z + \cdots)\left(\frac{1}{z^2} + \frac{2\gamma}{z} + \gamma^2\right)e^{zx}dz. \end{aligned}$$

Hence, retaining in the integrand only the terms with z^{-1} we obtain:

$$h(e^{-e^{-x}}) \approx \frac{e^x}{2\pi i}\oint_{z=0}\frac{1}{2z}(\gamma + x + \log 2)dz = \frac{e^x}{2}(\gamma + x + \log 2), \qquad (4)$$

which is the required asymptotic approximation for (3) when $x \to \infty$. Returning to the original geometrical variables where now $I \ll E$, we obtain:

$$e^{-e^{-x}} = \frac{E - I}{E + I} \approx 1 - \frac{2I}{E};$$

hence

$$e^{-x} = -\log\left(1 - \frac{2I}{E}\right) \approx \frac{2I}{E}, \qquad \text{or} \qquad x \approx \log\frac{E}{2I}.$$

Substitution in (4) yields

$$h(\alpha) \approx \frac{1}{2}\cdot\frac{E}{2I}\left(\gamma + \log\frac{E}{2I} + \log 2\right) = \frac{E}{4I}\left(\gamma + \log\frac{E}{I}\right);$$

hence we find as asymptotic expression for the limiting case where $E \gg I$ (spheres very close to each other)

$$C_{a,b} \approx \frac{E^2}{4c}\left(\gamma + \log\frac{E}{I}\right). \qquad (5)$$

This expression can be simplified still further if we introduce the small distance d between the spheres, where

$$d = c - (a + b).$$

Herewith we have approximately

$$E^2 \approx 4ab,$$
$$I^2 \approx 2d(a + b),$$

so that

$$C_{a,b} \approx \frac{ab}{a + b}\left(\gamma + \tfrac{1}{2} \log \frac{2ab}{d(a + b)}\right). \tag{6}$$

Again, if we assume the two spheres to be of equal size $(a = b)$, (6) becomes

$$C_{a,a} \approx \frac{a}{2}\left(\gamma + \tfrac{1}{2} \log \frac{a}{d}\right).$$

When the distance d is so small that we can even neglect γ with respect to $\frac{1}{2} \log \frac{a}{d}$, our final expression for $C_{a,a}$, the capacitance between two spheres of equal radius a, which are a very small distance d apart, becomes

$$C_{a,a} \approx \tfrac{1}{4}a \log \left(\frac{a}{d}\right).$$